# Announcements

1) Office hours change; no office hours Tuesday, office hours Thursday 9-11 AM

2) Practice problems for Exam 1 available under "Assignments" on Canvas.

# The Wronskian and Linear Independence

If $f, g$ are defined on an interval $I$ and there is a $t_0$ in $I$ with

$$\underbrace{\boxed{f'(t_0)g(t_0) - g'(t_0)f(t_0) = 0}}_{\text{(numerator of the quotient rule for } f/g\text{)}}$$

then $f$ and $g$ are linearly dependent on $I$.

Complex Numbers and Linear, homogeneous, second order equations or

what if the roots aren't real? (Section 4.3)

(Not on Exam 1)

# The Heat Equation

Temperature $u(x,t)$ at position $x$ and time $t$ in a thin rod given by

$$\boxed{\frac{\partial u}{\partial t}(x,t) = k \frac{\partial^2 u}{\partial x^2}(x,t)}$$

where $k \geq 0$ is the "thermal conductivity" of the material

## Wishful Thinking:

I wish that

$$u(x,t)=f(x)g(t)$$

for some real-valued functions $f$ and $g$.

Rewrite the heat equation.

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(f(x)g(t)\right)$$

$$= f(x)\,g'(t)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left(f(x)g(t)\right)$$

$$= f''(x)g(t)$$

( $t$ is a constant with respect to $x$,
$x$ is a constant with respect to $t$ )

The heat equation becomes

$$\boxed{f(x)\,g'(t) = k\,f''(x)\,g(t)}$$

Dividing both sides by $k\,f(x)\,g(t)$ we get

$$\frac{1}{k}\,\frac{g'(t)}{g(t)} = \frac{f''(x)}{f(x)}$$

Since $x$ and $t$ are independent variables, the only way a function of $x$ can equal a function of $t$ is if they are both constant!

So there is a number $\alpha$ with

$$\boxed{\frac{1}{k}\frac{g'(t)}{g(t)} = \frac{f''(x)}{f(x)} = \alpha}.$$

2 equations to solve:

1) $\frac{1}{k} \frac{g'(t)}{g(t)} = \alpha$.

Multiply by k to get

$\frac{g'(t)}{g(t)} = \alpha k$, integrate

with respect to t. We get

$\ln g(t) = \alpha k t + C$, so

Exponentiating,

$$\boxed{g(t) = e^{\alpha k t + C}}$$

2) $$\frac{f''(x)}{f(x)} = \alpha, \text{ so}$$

$$f''(x) = \alpha f(x) \text{ and}$$

$$f''(x) - \alpha f(x) = 0$$

We know how to solve!

Suppose $f(x) = e^{rx}$.

Then $f''(x) = r^2 e^{rx}$, so

we get

$$e^{rx}(r^2 + \alpha) = 0 \text{ and}$$

$$r^2 = \alpha, \text{ so}$$

$$r = \pm\sqrt{\alpha}.$$

Q: What if $\alpha < 0$?

Are there still solutions when $r$ is not a real number?

If $\alpha < 0$, then $-\alpha > 0$..

If we let $\boxed{f(x) = \sin(\sqrt{-\alpha}\,x)}$

$f''(x) = -\alpha(-\sin(\sqrt{-\alpha}\,x))$, so

$$f''(x) - \alpha f(x)$$

$$= -\alpha\left(-\sin(\sqrt{-\alpha}\,x) + \sin(\sqrt{-\alpha}\,x)\right)$$

$$= 0$$

So there are still solutions if $\alpha < 0$!

Could also find a solution in cosines: $f(x) = \cos(\sqrt{-\alpha}\,x)$

Since our original solutions were supposed to be

$e^{rx}$ where $r = \pm\sqrt{\alpha}$,

there should be a connection between $e^{rx}$, $\cos(\sqrt{-\alpha}\, x)$, and $\sin(\sqrt{-\alpha}\, x)$.

$\alpha < 0$ gives $r$ imaginary!

If $i = \sqrt{-1}$,

$r = \pm\sqrt{\alpha}$, $\alpha < 0$,

$$\boxed{r = \pm i\sqrt{-\alpha}}$$

Connection: MacLaurin Series!

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Next time: Calculate

$e^{ix}$ for $x \in \mathbb{R}$!